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A Characterization of Automata and a Direct Product Decomposition

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The classes of automata characterized by certain semigroups are investigated: It is shown that the classes of cyclic quasi-state-independent automata, cyclic quasi-state-independent automata of monoid type, cyclic Abelian automata, strongly connected state-independent automata, strongly connected reset automata, quasi-perfect automata, and perfect automata are equivalent to the classes of automata generated by semigroups with left identity, monoids, commutative semigroups with identity, right groups, right zero semigroups, groups, and Abelian groups, respectively. The characterization of the endomorphism semigroups and the automorphism groups and the direct product decomposabilities for the above classes of automata are also given. Finally, it is shown that every regular set can be accepted by some cyclic quasi-state-independent acceptor of monoid type.

1. INTRODUCTION

The homomorphism of an automaton preserves the state transition and it is also called an operation- (or structure-) preserving function. Then the endomorphism semigroup and the automorphism group of an automaton should reflect the structure of the given automaton, although this representation is not always perfect because an automorphism group is frequently a trivial group of identity. On the other hand, a state transition behavior of an automaton induces certain equivalence classes of its input semigroup which is also considered to reflect an algebraic feature of the structure of an automaton as well as the endomorphism semigroup and the automorphism group.

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In this sense, it is natural to study the relationship between such algebraic features of an automaton and the structure of an automaton or, from another standpoint, the relationship among the endomorphism semigroup, the automorphism group, the equivalence classes of its input semigroup, and other algebraic features. For the former, Fleck [1] investigated a class of perfect automata and showed that the direct product decomposability of a member of this class is equivalent to that of its automorphism group. Trauth [2] generalized Fleck's results, introducing a class of quasi-perfect automata and showing that similar conditions hold for the direct product decomposability of quasi-perfect automata. Other generalizations are also made by Fleck [3] and by Masunaga *et al.* [4]. One of the main purposes of this paper is to extend these discussions to the classes of automata which are characterized by certain semigroups and this is done in Section 5. For the latter, Weeg [5] showed that the automorphism group of a strongly connected automaton is isomorphic to a group of equivalence classes of its input semigroup. Some of the results given by him were generalized to cyclic automata by Oehmke [6], Arbib [7], and the converse problem of Weeg was investigated by Barnes [8]. The generalization of these discusssions is also one of the main purposes of this paper and this is done in Section 4. The fundamental considerations are developed in Section 3, and this is the essential part of this paper. Several classes of automata which are characterized by certain semigroups are investigated in terms of an automaton generated by a semigroup. The results are as follows. The classes of cyclic quasi-state-independent automata, cyclic quasi-state-independent automata of monoid type, cyclic Abelian automata, strongly connected state-independent automata, strongly connected reset automata, quasi-perfect automata, and perfect automata are equivalent to the classes of automata generated by semigroups with left identity, monoids, commutative semigroups with identity, right groups, right zero semigroups, groups, and Abelian groups, respectively. Finally, it is shown in Section 6 that every regular set can be accepted by some cyclic quasi-state-independent acceptor of monoid type.

2. Preliminaries

An automaton is a 3-tuple $A = (Q, M, I)$, where Q is a finite nonempty set of states, I is an input semigroup, and $M: Q \times I \to Q$ is a state transition function such that $\forall q \in Q$, $\forall x, y \in I$, $M(q, xy) = M(M(q, x), y)$.

In this paper, all automata are assumed to be complete; i.e., M is defined for any (q, x) in $Q \times I$. But the existence of an identity element in I is not always assumed because sometimes the existence of it restricts structure of an automaton (see Proposition 3.7). An automaton is called cyclic if there exists a generator q_0 of it such that $\forall q \in Q$, $\exists x \in I$, $q = M(q_0, x)$. An automaton is called strongly connected if every state is a generator of it.

Let $A = (Q, M, I)$ and $B = (R, N, I)$ be automata. A mapping $\eta: Q \to R$ is called a homomorphism of A into B if it satisfies the condition

$$\forall q \in Q, \qquad \forall x \in I, \qquad \eta(M(q, x)) = N(\eta(q), x).$$

If η is an onto mapping satisfying the above condition, it is called a homomorphism of A onto B. η is called an isomorphism of A into B if it is a one-to-one mapping. A is said to be isomorphic to B if there exists an isomorphism of A onto B. Especially, a homomorphism of A into itself is called an endomorphism of A and an isomorphism of A onto itself is called an automorphism of A. By $E(A)$ and $G(A)$, we denote the set of all endomorphisms of A and the set of all automorphisms of A, respectively. It is well known that $E(A)$ forms a semigroup with an identity and $G(A)$ foıms a group under the usual functional composition, respectively [1].

It is also known that certain equivalence classes of an input semigroup, which is introduced by Weeg [5] as the input semigroup associated with an automaton (the same notion is also introduced by Krohn and Rhodes [9] as the semigroup of a machine), represent some algebraic features of the structure of an automaton. It is defined as follows. Let $A = (Q, M, I)$ be an automaton and define relations ρ_q $(q \in Q)$ and ρ_A such that $\forall x,\ y \in I,\ x \rho_q y \Leftrightarrow M(q, x) = M(q, y)$ and $\rho_A = \bigcap_{q \in Q} \rho_q$. It is easily proved that ρ_A is a congruence relation on I and ρ_q is a right congruence relation on I. Notice that ρ_q is not always a congruence relation on I. Then a quotient semigroup of I modulo ρ_A, denoted by $\bar{I}(A)$, is defined as follows. $\forall [x]_A, [y]_A \in \bar{I}(A)$, $[x]_A \circ [y]_A = [xy]_A$ where $\circ$ denotes a natural product operation of $\bar{I}(A)$ and $[x]_A$ denotes an equivalence class of ρ_A containing x. ρ_A is equal to ρ_B if A is isomorphic to B. The relationship among $\bar{I}(A)$, $E(A)$, and $G(A)$ of an automaton A has been discussed by Weeg [5], Fleck [1, 3], Oehmke [6], Arbib [7], Barnes [8], Trauth [2], and many other authors.

3. A Characterization of Automata

Let ρ be a right congruence relation on a semigroup I. Then we can define an automaton $A(I/\rho)$ which is called an automaton induced by a right congruence relation ρ on the input semigroup I as follows. $A(I/\rho) = (I/\rho, M_\rho, I)$, $\forall [x] \in I/\rho, \forall y \in I$, $M_\rho([x], y) = [xy]$, where $[x]$ denotes an equivalence class of ρ containing x in I. As is well known, I/ρ does not form a semigroup generally when ρ is a right congruence relation. It forms a semigroup under the natural product operation named a quotient semigroup of I modulo ρ, if and only if ρ is a congruence relation. Our major concerns lie in this case. That is, if ρ is a congruence relation, then the state transition structure of $A(I/\rho)$ is completely determined by the semigroup operation of I/ρ.

Definition 3.1. Let ρ be a congruence relation on the input semigroup I. Then $A(I/\rho)$ is called an automaton generated by a semigroup I/ρ.

Here we should remark that the automaton generated by a semigroup is essentially the same as "the canonical machine" of Krohn and Rhodes [9] and Zeiger [10] and "the state-machine of semigroup accumulator" of Hartmanis and Stearns [11].

Remark 3.1. Strictly speaking, Definition 3.1 and the definitions of [9, 10, 11] are equivalent if we slightly extend the isomorphism of automata as follows (cf. [11]). An automaton $A = (Q, M, I)$ is isomorphic to $B = (R, N, J)$ if there exists a pair of mappings (η, δ) which satisfies $\eta(M(q, x)) = N(\eta(q), \delta(x))$, where η is a one-to-one mapping of Q onto R and δ is a semigroup homomorphism of I onto J. In our case, $A(I/\rho) = (I/\rho, M_\rho, I)$ is isomorphic to $B = (I/\rho, M_{I/\rho}, I/\rho)$ because $(\iota, \rho^\natural)$ satisfies the isomorphism, where ι is an identity mapping and $\rho^\natural$ is the natural homomorphism of I onto $I/\rho^\natural$ induced by ρ.

Now, let us restrict our discussion on the structure of cyclic automata. As is well known, a cyclic automaton $A = (Q, M, I)$ with a generator q_0 is isomorphic to $A(I/\rho_{q_0})$, which is the automaton induced by ρ_{q_0}, under the isomorphism: $q \leftrightarrow [x]_{q_0}$, where $q = M(q_0, x)$ and $[x]_{q_0}$ denotes an equivalence class of ρ_{q_0} containing x. So, if ρ_{q_0} forms a congruence relation, then the (state transition) structure of the given cylic automaton can be determined by the structure of the semigroup I/ρ_{q_0}. The classes of automata investigated in this paper are the subclasses of the class of this type of automata. To advance our discussion, we shall introduce "quasi-state-independentness" of automata, which is a generalization of "state-independentness" by Trauth [2].

DEFINITION 3.2. An automaton $A = (Q, M, I)$ is called quasi-state-independent with respect to a state q if it satisfies the condition

$$\forall q' \in Q, \qquad \rho_q \subseteqq \rho_{q'} \,.$$

Remark 3.2. Suppose a cylic automaton A is quasi-state-independent with respect to a generator q_0. Then $\rho_{q_0} = \rho_{q_0'}$ for any other generator q_0' of A. Therefore we shall simply say that a cyclic automaton is quasi-state-independent when it is quasi-state-independent with respect to some generator of it.

Now, the next results are immediate.

PROPOSITION 3.1. *Let A be a cyclic quasi-state-independent automaton. Then* $A \simeq A(\bar{I}(A))$.

PROPOSITION 3.2. *Let A be a cyclic quasi-state-independent automaton. Then $\bar{I}(A)$ forms a semigroup with a left identity.*

Proof. ρ_A ($= \rho_{q_0}$, q_0 is a generator of A) is a modular congruence relation. Q.E.D.

Here, we can show a characterization theorem for the class of cyclic quasi-state-independent automata, which is essential in the following discussion.

THEOREM 3.1. *An automaton is cyclic and quasi-state-independent if and only if it is isomorphic to an automaton generated by a semigroup with a left identity.*

Proof. The "only if" part is true by Propositions 3.1 and 3.2. Conversely, let I/ρ be a semigroup with a left identity e. Then e is a generator of $A(I/\rho)$ and $A(I/\rho)$ is quasi-state-independent because $x \rho_e y \Leftrightarrow [x] = [y] \Rightarrow \forall z \in I, [zx] = [zy] \Leftrightarrow \forall z \in I, x \rho_{[z]} y$, where $[x]$ denotes an equivalence class of ρ containing x. Q.E.D.

We shall remark on another aspect of quasi-state-independentness.

Remark 3.3. Let $A = (Q, M, I)$ be a cyclic automaton with a generator q_0. Weeg [5] defined an operation $*_{q_0}$ on the set I/ρ_{q_0} such that $[x]_{q_0} *_{q_0} [y]_{q_0} = [z]_{q_0}$ if $xy \rho_{q_0} z$. But as he mentioned, $*_{q_0}$ is not always well defined. In response, we say that $*_{q_0}$ is well defined if and only if A is quasi-state-independent.

Now, we shall continue our discussion, introducing "state-independentness" of automata defined by Trauth [2].

DEFINITION 3.3. An automaton $A = (Q, M, I)$ is called state-independent if it satisfies the condition

$$\forall q, r \in Q, \qquad \rho_q = \rho_r .$$

It is clear that if an automaton is state-independent, then it is quasi-state-independent with respect to any state of it. The next proposition states a relation among the three considerable classes of automata. The proof is direct and is therefore omitted.

PROPOSITION 3.3. *The following three assertions concerning an automaton A are equivalent.*

(i) *A is a cyclic state-independent automaton.*

(ii) *A is a strongly connected quasi-state-independent automaton.*

(iii) *A is a strongly connected state-independent automaton.*

So it is possible to restrict our discussion to characterize strongly connected state-independent automata. We must first show the next proposition.

PROPOSITION 3.4. *Let A be a strongly connected state-independent automaton. Then $\bar{I}(A)$ forms a right group.*

Proof. A finite semigroup is a right group if and only if it is right simple (or equivalently, left cancellative). Because A is state-independent, ρ_q $(= \rho_A)$ is a congruence relation for any q in Q. So, $\bar{I}(A) = I/\rho_A = I/\rho_q$ is right simple if ρ_q is transitive. But this is true because A is strongly connected. Q.E.D.

The next is a characterization theorem for the class of strongly connected state-independent automata.

THEOREM 3.2. *An automaton is strongly connected state-independent if and only if it is isomorphic to an automaton generated by a right group.*

Proof. We shall show the "if" part. An automaton generated by a right group is strongly connected and state-independent because of the right simplicity and left cancellativity, respectively, of a right group. Q.E.D.

In Section 5, another characterization will be given for this class of automata (Corollary 5.6). Here we shall continue our discussion, introducing reset automata.

DEFINITION 3.4. $A = (Q, M, I)$ is called a reset automaton if it satisfies the condition

$$\forall x, y \in I, \qquad \forall q \in Q, \qquad M(q, xy) = M(q, y).$$

It is clear that A is a reset automaton if and only if $\forall x, y \in I$, $\forall q \in Q$, $xy\ \rho_q\ y$.

PROPOSITION 3.5. *Let A be a cyclic reset automaton. Then A is strongly connected and state-independent.*

Proof. Let q_0 be a generator of A. Then $\forall x$, $y \in I$, $(x\ \rho_{q_0}\ y \Leftrightarrow \forall z \in I$, $zx\ \rho_{q_0}\ (x\ \rho_{q_0}\ y\ \rho_{q_0})zy)$ and $\forall x$, $y \in I$, $xy\ \rho_{q_0}\ y$ (therefore ρ_{q_0} is a transitive congruence relation) implies A is strongly connected state-independent. Q.E.D.

So the next result is immediate.

PROPOSITION 3.6. *Let A be a strongly connected reset automaton. Then $\bar{I}(A)$ forms a right zero semigroup.*

Here we can show a characterization theorem for the class of strongly connected reset automata.

THEOREM 3.3. *An automaton is a strongly connected reset automaton if and only if it is isomorphic to an automaton generated by a right zero semigroup.*

Proof. We shall show the "if" part. Suppose I/ρ forms a right zero semigroup. Then $A(I/\rho)$ is strongly connected because every element of I/ρ is a left identity for ρ, and is a reset automaton because $xy\ \rho\ y$ for all x and y. Q.E.D.

Trauth [2] introduced a class of quasi-perfect automata as follows.

DEFINITION 3.5. An automaton is called quasi-perfect if it is strongly connected state-independent and the input semigroup associated with it forms a group.

Because the input semigroup associated with a strongly connected state-independent automaton forms a right group (Proposition 3.4), the class of quasi-perfect automata forms a proper subclass of that of strongly connected state-independent automata. The next is a characterization theorem for the class of quasi-perfect automata. The proof is direct and is therefore omitted.

THEOREM 3.4. *An automaton is quasi-perfect if and only if it is isomorphic to an automaton generated by a group.*

Now we should examine the case where the input semigroup has an identity element because of the fact that a right group with an identity is indeed a group. Bayer [12] showed that a strongly connected state-independent automaton is equivalent to a quasi-perfect automaton. But this is true because his input semigroup always has an identity element.

PROPOSITION 3.7. *Let A be an automaton with an identity in its input semigroup. Then the following two assertions concerning A are equivalent.*

(i) *A is a strongly connected state-independent automaton.*

(ii) *A is a quasi-perfect automaton.*

Here we should note the equivalence of a variety of concepts on quasi-perfect automate investigated by many authors. The analogous result has already been found in [12].

PROPOSITION 3.8. *Let A be an automaton. Then the following* 12 *assertions concerning A are equivalent* (*assertions* (10) *to* (12) *are equivalent to the others in the sense of Remark* 3.1).

(1) *A is a quasi-perfect automaton of Trauth* [2].

(2) *A is a strongly connected automaton with a transitive automorphism group* [2].

(3) *A is a total automaton of Bayer* [12].

(4) *An automaton A satisfying* [13, *Theorem* 1].

(5) *An automaton A satisfying the conditions of* [8, *Theorem* 3].

(6) *An automaton A satisfying* [3, *Theorem* 2.2].

(7) *An automaton A constructed by Corollary of* [14, *Theorem* 10].

(8) *A is a group automaton of Arbib* [7].

(9) *A is an automaton generated by a group in this paper.*

(10) *A is a canonical machine of group of Krohn and Rhodes* [9].

(11) *A is a state machine of a group accumulator of Hartmanis and Stearns* [11].

(12) *A is a regular automaton of a group of Deussen* [15].

We shall proceed to characterize other classes of automata.

DEFINITION 3.6. An automaton $A = (Q, M, I)$ is called Abelian if it satisfies the condition

$$\forall q \in Q, \qquad \forall x, y \in I, \qquad M(q, xy) = M(q, yx).$$

PROPOSITION 3.9. *Let A be a cyclic Abelian automaton. Then A is quasi-state-independent.*

Proof. Let q_0 be a generator of A. Then $\forall x, y \in I$, $(x \,\rho_{q_0}\, y \Rightarrow (\forall z \in I,\ xz \,\rho_{q_0}\, yz \Rightarrow zx \,\rho_{q_0}\, zy))$ implies A is quasi-state-independent. Q.E.D.

So the next result is immediate.

PROPOSITION 3.10. *Let A be a cyclic Abelian automaton. Then $\bar{I}(A)$ forms a commutative semigroup with an identity.*

The next is a characterization theorem for the class of cyclic Abelian automata. The proof is straightforward and is omitted here.

THEOREM 3.5. *An automaton is cyclic Abelian if and only if it is isomorphic to an automaton generated by a commutative semigroup with an identity.*

The class of perfect automata introduced by Fleck [1] is a special subclass of the class of cyclic Abelian automata. We shall first show the definition of perfect automata.

DEFINITION 3.7. An automaton is called perfect if it is strongly connected and Abelian.

It is known that the class of perfect automata forms a proper subclass of that of quasi-perfect automata; i.e., an automaton is perfect if and only if it is quasi-perfect and the input semigroup associated with it forms an Abelian group [2]. So the next theorem is obtained.

THEOREM 3.6. *An automaton is perfect if and only if it is isomorphic to an automaton generated by an Abelian group.*

We have characterized several classes of automata which are, as a result, equivalent to certain classes of automata generated by semigroups or groups. But we can also characterize other classes of automata in terms or homomorphisms of automata. For example, an automaton is a strongly connected permutation automaton if and only if it is a homomorphic image of a quasi-perfect automaton [4]. The reader should note that the input semigroup associated with a cyclic automaton forms a left reductive semigroup. So this is a weak characterization to say that an automaton is cylic only if it is a homomorphic image of an automaton generated by a left reductive semigroup. The converse of this result is not always true.

4. Endomorphism Semigroups and Automorphism Groups of Automata

In this section, we shall characterize the endomorphism semigroup and the automorphism groups of automata investigated in the previous section. The fundamental characterizations of the endomorphism semigroup and the automorphism group of a cyclic automaton (then for a strongly connected automaton) were given in [6; 7; 20, Chap. 11], using the normalizer of a right congruence relation on a semigroup. The next is one of the main results proposed by them. Let A be a cyclic automaton with a generator q_0. As pointed out in Section 2, ρ_{q_0} is a right congruence relation on an input semigroup I. Then the normalizer $N(\rho_{q_0})$ of ρ_{q_0} on I is defined as follows. $N(\rho_{q_0}) = \{y \in I \mid \forall x_1, x_2 \in I, x_1 \rho_{q_0} x_2 \Rightarrow yx_1 \rho_{q_0} yx_2\}$. $N(\rho_{q_0})$ forms a subsemigroup of I and a union of ρ_{q_0}-classes. Next let x_0 be an element of I such that $M(q_0, x_0) = q_0$. Then ρ_{q_0} is a modular right congruence relation on I because $\forall x \in I$, $x_0 x \rho_{q_0} x$. Now define $N_{x_0}(\rho_{q_0}) = \{y \in N(\rho_{q_0}) \mid yx_0 \rho_{q_0} y\}$. Clearly $N_{x_0}(\rho_{q_0})$ is a left ideal $N(\rho_{q_0})$ and a union of ρ_{q_0}-classes. $N_{x_0}(\rho_{q_0})/\rho_{q_0}$ is the principal left ideal of $N(\rho_{q_0})/\rho_{q_0}$ generated by the idempotent $[x_0]_{q_0}$, where $[x_0]_{q_0}$ denotes an equivalence class of ρ_{q_0} containing x_0.

Proposition 4.1. *Let A be a cyclic automaton with a generator q_0. Then $E(A)$ is isomorphic to $N_{x_0}(\rho_{q_0})/\rho_{q_0}$. Here the isomorphism ι of $N_{x_0}(\rho_{q_0})/\rho_{q_0}$ onto $E(A)$ is defined as follows. $\iota([y]_{q_0}) = \lambda_y$ where λ_y is a mapping such that $\lambda_y(q) = M(q, yx)$ for any q $(= M(q_0, x))$ in Q.*

By this proposition, the next result concerning a cylic quasi-state-independent automaton is clear since $N(\rho_{q_0}) = I$ in this case.

Corollary 4.1. *Let A be a cyclic quasi-state-independent automaton with a generator q_0. Then $E(A)$ is isomorphic to the principal left ideal of $\bar{I}(A)$ generated by $[x_0]_A$, where $M(q_0, x_0) = q_0$.*

We continue to discuss a relation between $\bar{I}(A)$ and $E(A)$ in the cyclic quasi-state-independent case. Under the same condition of this corollary, it is clear that $N_{x_0}(\rho_{q_0}) = I$ if and only if x_0 is a right identity of I modulo ρ_{q_0}. In this case $[x_0]_A$ is an identity of $\bar{I}(A)$.

Corollary 4.2. *Let A be a cyclic quasi-state-independent automaton. Then $E(A)$ is isomorphic to $\bar{I}(A)$ if and only if $\bar{I}(A)$ forms a monoid.*

By this corollary and Proposition 3.10 the next result is obvious.

COROLLARY 4.3. *Let A be a cyclic Abelian automaton. Then $E(A)$ is isomorphic to $\bar{I}(A)$.*

But Corollary 4.2 is a weak characterization of the endomorphism semigroup of a cyclic quasi-state-independent automaton in the following sense. We can show a stronger result by introducing a few definitions and discussions.

DEFINITION 4.1. An automaton A is called of monoid type if $\bar{I}(A)$ forms a monoid.

DEFINITION 4.2. Let A be an automaton. A is said to be transitive with respect to $E(A)$ and a state q if it satisfies the condition

$$\forall q' \in Q, \qquad \exists h \in E(A), \qquad q' = h(q).$$

We shall consider the case when an automaton is cyclic.

PROPOSITION 4.2. *Let A be a cyclic automaton with a generator q_0. If A is transitive with respect to $E(A)$ and q_0, then A is transitive with respect to $E(A)$ and any generator of A.*

Proof. The proof is direct by using the fact that for $h, h' \in E(A)$, $h \neq h'$ if and only if $h(q) \neq h'(q)$ for any generator q of A. Q.E.D.

By this proposition we shall simply say that an automaton is transitive with respect to its endomorphism semigroup when it is transitive with respect to its endomorphism semigroup and some generator of it. The next is a desired result.

THEOREM 4.1. *Let A be a cyclic automaton. Then the following four assertions concerning A are equivalent.*

(i) *$E(A)$ is isomorphic to $\bar{I}(A)$.*

(ii) *The order of $E(A)$ is equal to the total number of states of A.*

(iii) *A is transitive with respect to $E(A)$.*

(iv) *A is a quasi-state-independent automaton of monoid type.*

The automaton which satisfies the above conditions is isomorphic to an automaton generated by some monoid.

Proof. (i) $\to$ (ii): The order of $E(A)$ is usually less than or equal to the total number of states and the order of $\bar{I}(A)$ is usually greater than or equal to the total of states when an automaton A is cyclic. (ii) $\to$ (iii): For $h, h' \in A(E)$, $h = h'$ if and only if there exists a generator q_0 of A such that $h(q_0) = h'(q_0)$. (iii) $\to$ (iv): A is quasi-state-independent because $\forall h \in E(A)$, $x \rho_{q_0} y \Rightarrow x \rho_{h(q_0)} y$, where q_0 is a generator of A. Let $x_0 \in I$ be such that $M(q_0, x_0) = q_0$. Then $[x_0]_A$ is an identity of $\bar{I}(A)$ because A is transitive with respect to $E(A)$. (iv) $\rightarrow$ (i): By Corollary 4.2. The last assertion can be shown directly by Theorem 3.1. Q.E.D.

It is clear, then, that the structure of a member of the class of cyclic quasi-state-independent automata of monoid type can be completely determined by its endomorphism semigroup. Moreover, this theorem is a nice extension of the result characterizing the automorphism groups of quasi-perfect automata, which will be mentioned again at the end of this section (Proposition 4.5). This type of automata will also play an essential role in Section 6.

Now we proceed to characterize the endomorphism semigroups and the automorphism groups of strongly connected state-independent automata and some other subclasses of them.

PROPOSITION 4.3. *Let A be a strongly connected state-independent automaton. The $E(A)$ is identical to $G(A)$ and it is isomorphic to the principal left ideal of $\bar{I}(A)$ generated by $[x_0]_A$, where $M(q, x_0) = q$ for some q in Q.*

Proof. In this case, the principal left ideal of $\bar{I}(A)$ generated by $[x_0]_A$ forms a group because $\bar{I}(A)$ is a right group (Proposition 3.4). Then $E(A)$ must be a group by Corollary 4.1. Q.E.D.

Because $\bar{I}(A)$ forms a right group when A is a strongly connected state-independent automaton, we can give a more detailed relation between $G(A)$ and $\bar{I}(A)$ using a well-known result concerning a right group: Let Z be a set of all idempotent elements of a right group $\bar{I}(A)$. Then $\bar{I}(A)$ is isomorphic to a direct product of Z and the principal left ideal generated by a left identity element of $\bar{I}(A)$. Notice that Z forms a right zero semigroup and it is constructed by $Z = \{[x]_A \in \bar{I}(A) \mid \exists q \in Q, M(q, x) = M(q, x^2)\}$.

PROPOSITION 4.4. *Let A be a strongly connected state-independent automaton. Then $\bar{I}(A)$ is isomorphic to a direct product of $G(A)$ and a right zero semigroup of all idempotent elements of $\bar{I}(A)$.*

This result will be used in the next section.

COROLLARY 4.4. *Let A be a strongly connected reset automaton. Then $E(A)$ is identical to $G(A)$ and it forms a trivial group of identity.*

Proof. By Propositions 3.6 and 4.3. Q.E.D.

At the end of this section, we must take note of some of the results for the characterization of the automorphisms of quasi-perfect automata given by Fleck [1], Trauth [2], and Bayer [12]. Since the principal left ideal of a group is identical to the given group, $E(A)$ is identical to $G(A)$ and it is isomorphic to $\bar{I}(A)$ when an automaton A is quasi-perfect. But the next stronger result should be compared with Theorem 4.1.

Here we shall use the terminologies introduced by Trauth: An automaton A is called of group type if $\bar{I}(A)$ forms a group and A is said to be transitive with respect to $G(A)$ if it satisfies the condition

$$\forall q, q' \in Q, \qquad \exists g \in G(A), \qquad q' = g(q).$$

PROPOSITION 4.5. *Let A be a strongly connected automaton. Then the following four assertions concerning A are equivalent.*

(i) *$G(A)$ is isomorphic to $\bar{I}(A)$.*

(ii) *The order of $G(A)$ is equal to the number of states of A.*

(iii) *A is transitive with respect to $G(A)$.*

(iv) *A is a state-independent automaton of group type.*

The automaton which satisfies the above conditions is isomorphic to an automaton generated by some group (Theorem 3.4).

5. DIRECT PRODUCT DECOMPOSITION OF AUTOMATA

Let $A = (Q, M, I)$ and $B = (R, N, I)$ be automata. A direct product of A and B, denotes by $A \times B$, is an automaton defined as follows [16].

$A \times B = (Q \times R, \overline{M}, I)$, where $\overline{M}((q, r), x) = (M(q, x), N(r, x))$ for any $(q, r) \in Q \times R$ and $x \in I$. An automaton A is said to be decomposable into a direct product of two (factor) automata B and C if A is isomorphic to $B \times C$. It is known that an automaton A is decomposable into a direct product of two automata if and only if there exist two automaton congruences π_1 and π_2 of A such that $\pi_1 \cap \pi_2 = 0$ (identity relation) and $\pi_1 \,\square\, \pi_2 = 1$ (universal relation) [17].

A direct product of two semigroups S and T, denoted by $S \times T$, is defined in the usual fashion and a semigroup S is said to be decomposable into a direct product of two (factor) semigroups T and U if S is isomorphic to $T \times U$. It is easily proved that a semigroup S is decomposable into a direct product of two semigroups if and only if there exist two congruence relations τ_1 and τ_2 on S such that $\tau_1 \cap \tau_2 = 0$ and $\tau_1 \,\square\, \tau_2 = 1$.

To obtain our desired result, we shall describe the direct product decomposability of an automaton generated by a semigroup.

PROPOSITION 5.1. *Let ρ be a congruence relation on I. Then $A(I/\rho)$ is decomposable into a direct product of two automata if I/ρ is decomposable into a direct product of two semigroups. In this case, each factor automaton is also generated by some semigroup.*

Proof. Suppose $I/\rho \cong S \times T$. Then there exist two congruence relations γ and δ on I such that $S \cong I/\gamma$ and $T \cong I/\delta$. So, $I/\rho \cong I/\gamma \times I/\delta$ implies $\gamma \cap \delta = \rho$ and $\gamma \sqcup \delta = 1$.

Then $A(I/\rho) \cong A(I/\gamma) \times A(I/\delta)$ is true because a mapping $\eta: I/\rho \to I/\gamma \times I/\delta$ defined by $\forall [x] \in I/\rho$, $\eta([x]) = ([x]_\gamma, [x]_\delta)$ is an isomorphism; i.e., η is one-to-one because $\gamma \cap \delta = \rho$. η is onto because $\gamma \sqcup \delta = 1$. η is directly determined to be a homomorphism. Q.E.D.

We should stress here that a direct decomposability of a semigroup, which generates an automaton, is only a sufficient condition for the generated automaton to be decomposable into a direct product of automata. This is true even for a quasi-perfect automaton, as the reader can easily find in [2]. But if we restrict our discussion to the case where every factor automaton should be also generated by some semigroup, then we can show a necessary and sufficient condition for direct product decomposability of automata generated by semigroups. We introduce left reductiveness of a congruence relation:

DEFINITION 5.1. Let ρ be a congruence relation on a semigroup S. Then ρ is called left reductive if $zx \,\rho\, zy$ for all z in S implies $x \,\rho\, y$.

It is clear that S/ρ forms a left reductive semigroup if and only if ρ is a left reductive congruence relation on S.

Now, let ρ be a congruence relation on I. Then it is easily proved that $\rho \subseteq \rho_{A(I/\rho)}$ and the converse inclusion relation is not always true.

PROPOSITION 5.2. *Let ρ be a congruence relation on I. Then $\rho = \rho_{A(I/\rho)}$ if and only if ρ is left reductive.*

Proof. Suppose $x \,\rho_{A(I/\rho)}\, y$ for x, y in I; i.e., $\forall [z] \in I/\rho$, $[z] \circ [x] = [z] \circ [y]$. But this formula is equivalent to $[x] = [y]$, i.e., $x \,\rho\, y$, if and only if ρ is left reductive. Q.E.D.

COROLLARY 5.1. *Let ρ be a congruence relation on I. Then $\bar{I}(A(I/\rho)) = I/\rho$ if and only if ρ is left reductive.*

The results indicate that when an automaton is generated by a left reductive semigroup, we can identify the input semigroup associated with it with the semigroup which generates the automaton. Let us continue our discussion.

THEOREM 5.1. *Let ρ, γ, δ be left reductive congruence relations on I. Then $A(I/\rho) \cong A(I/\gamma) \times A(I/\delta)$ if and only if $I/\rho \cong I/\gamma \times I/\delta$.*

Proof. Suppose $A(I/\rho) \cong A(I/\gamma) \times A(I/\delta)$ under the isomorphism $[x] \leftrightarrow ([x]_\gamma, [x]_\delta)$. Then $\rho = \gamma \cap \delta$ because $x \,\rho\, y \Leftrightarrow x \,\rho_{A(I/\rho)}\, y \Leftrightarrow x \,\rho_{A(I/\gamma)\times A(I/\delta)}\, y \Leftrightarrow x(\rho_{A(I/\gamma)} \cap \rho_{A(I/\delta)})y \Leftrightarrow$

$x(\gamma \cap \delta)y$, and $\gamma \sqcap \delta = 1$ because $\forall([x]_\gamma, [y]_\delta) \in I/\gamma \times I/\delta, \exists[z] \in I/\rho, [z] \leftrightarrow ([x]_\gamma, [y]_\delta)$. So $I/\rho = I/(\gamma \cap \delta) = I/\gamma \times I/\delta$. Q.E.D.

COROLLARY 5.2. *Let A be an automaton generated by a left reductive semigroup. Then A is decomposable into a direct product of two factor automata, and each factor automaton is also generated by some left reductive semigroup, if and only if $\bar{I}(A)$ is decomposable into two semigroups.*

Here we should note that these results are essential for characterizing the necessary and sufficient condition of direct product decomposability of automata introduced in the previous sections, because a semigroup with a left identity, a monoid, a commutative semigroup with an identity, a right group, right zero semigroup, a group, and an Abelian group are all left reductive semigroups. So we have many corollaries of this result, as follows.

COROLLARY 5.3. *A cyclic quasi-state-independent automaton A is decomposable into a direct product of two cyclic quasi-state-independent automata if and only if $\bar{I}(A)$ is decomposable into a direct product of two semigroups.*

COROLLARY 5.4. *A cyclic quasi-state-independent automaton A of monoid type is decomposable into a direct product of two cyclic quasi-state-independent automata of monoid type if and only if $E(A)$ is decomposable into a direct product of two semigroups.*

Proof. This is true by Corollaries 5.3 and 4.2. Q.E.D.

COROLLARY 5.5. *A cyclic Abelian automaton A is decomposable into a direct product of two cyclic Abelian automata if and only if $E(A)$ is decomposable into a direct product of two semigroups.*

Proof. This is true by Corollaries 5.2 and 4.3. Q.E.D.

COROLLARY 5.6. *Let A be a strongly connected state-independent automaton. Then A is isomorphic to a direct product of a quasi-perfect automaton and a strongly connected reset automaton.*

Proof. This is true by Corollary 5.2, Proposition 4.4, and Theorems 3.3 and 3.4. Q.E.D.

The next result was given by Trauth [2].

COROLLARY 5.7. *A quasi-perfect automaton A is decomposable into a direct product of two quasi-perfect automata if and only if $G(A)$ is decomposable into a direct product of two groups.*

PROPOSITION 5.3. *A strongly connected state-independent automaton A is isomorphic to a direct product of a quasi-perfect automaton B and a strongly connected reset automaton*

C. Then B is decomposable into a direct product of two quasi-perfect automata if and only if G(A) is decomposable into a direct product of two groups.

Proof. This is true by Corollaries 5.6 and 5.7 and Proposition 4.4. Q.E.D.

A direct product decomposability of perfect automaton is found in [1], a direct product decomposability of a quasi-perfect automaton is extended to a general case in [3], and a direct product decomposability of a strongly connected permutation automaton is given in [4].

6. Recognition Ability of Cyclic Quasi-State-Independent Acceptor of Monoid Type

In this section, we shall show that every regular set can be accepted by some cyclic quasi-state-independent acceptor of monoid type.

Definition 6.1. An acceptor is a 5-tuple $A = (Q, M, \Sigma^*, q_0, F)$, where Q is a nonempty set of states, Σ^* is a free monoid generated by a finite alphabet Σ, M: $Q \times \Sigma^* \to Q$ is a state transition function, q_0 is an initial state, and $F \subseteqq Q$ is a set of final states.

By $T(A)$ we denote a set of tapes accepted by A. $T(A)$ is defined as follows: $T(A) = \{t \in \Sigma^* \mid M(q_0, t) \in F\}$. An acceptor $A = (Q, M, \Sigma^*, q_0, F)$ is called of particular structure type if its semiautomaton [18] $A' = (Q, M, \Sigma^*)$ is of this structure type. For example an acceptor A is cyclic quasi-state-independent of monoid type if its semiautomaton A' is cyclic quasi-state-independent of monoid type. The desired result is as follows.

Theorem 6.1. *Let A be an acceptor. Then there exists a cyclic quasi-state-independent acceptor of monoid type B such that* $T(A) = T(B)$.

Proof. Let $A = (Q, M, \Sigma^*, q_0, F)$ be an acceptor and $A' = (Q, M, \Sigma^*)$ be its semiautomaton. It is natural to assume that A' is cyclic with an initial state q_0. Now, we shall construct a cyclic quasi-state-independent acceptor B of monoid type as follows. $B = (\bar{\Sigma}^*(A'), \bar{M}, \Sigma^*, [t_0], [t]_F)$, where $\bar{\Sigma}^*(A')$ is an input monoid associated with A', $\bar{M}([t], u) = [tu]$ for $[t] \in \bar{\Sigma}^*(A')$ and $u \in \Sigma^*$, $[t_0]$ is such that $M(q_0, t_0) = q_0$ and $[t]_F = \{[t] \in \bar{\Sigma}^*(A') \mid M(q_0, t) \in F\}$. $\bar{\Sigma}^*(A')$ is a monoid since Σ^* contains a null word as an identity. Then B is a cylic quasi-state-independent acceptor of monoid type because the semiautomaton of it is generated by the monoid $\bar{\Sigma}^*(A')$ (Theorem 4.1). $T(A) = T(B)$ because the semiautomaton of A is a homomorphic image of the semiautomaton of B under the homomorphism $[t] \to M(q_0, t)$ which does not change the acceptability of tapes by the definition of $[t]_F$. Q.E.D.

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